Name: Solution .
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Section: .

Exam II

Thursday December 22, 2011
Duration: 1h 30min
All work **must** be shown to receive full credit.

Problem 1. (15 pts.) A mass spring damper system with a mass $m = 10\ Kg$ and a damping ratio $\zeta = 0.3$ is subjected to a harmonic force $f(t) = 50\ sin(\omega t)\ N$. If the maximum amplitude is observed at $\omega = 60\ rad/s$ and the steady state solution takes the form $x(t) = X_{ss}\sin(60\,t - \phi)$, calculate X_{ss} and ϕ.

$m = 10\ kg$, $f_0 = 50\ N$, $\zeta = 0.3$, $\omega_{peak} = 60\ rad/s$

$$\omega_n = \frac{\omega_{peak}}{\sqrt{1-2\zeta^2}} = \frac{60}{\sqrt{1-2\times 0.3^2}} = 66.26\ rad/s$$

$$k = m\omega_n^2 = 10 \times 66.26^2 = 43902.4\ N/m$$

$$c = 2\zeta\sqrt{mk} = 2\times 0.3\sqrt{10\times 43902.4} = 397.55\ N.s/m$$

$$X_{ss} = \frac{f_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} = \frac{50}{\sqrt{(43902.4 - 10\times 60^2)^2 + (397.55\times 60)^2}}$$

$$X_{ss} = 0.00198\ m$$

$$\phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) = \tan^{-1}\left(\frac{397.55\times 60}{43902.4 - 10\times 60^2}\right) = 1.25\ rad$$

$$\Rightarrow x(t) = 0.00198\sin(60t - 1.25)\ m$$

Problem 2. (15 pts.) A heavy machine of mass $m = 200\ Kg$ is sitting on a resilient material (spring damper) with a stiffness of $k = 25000\ N/m$ and a damping constant of $c = 50\ Ns/m$. The machine generates a force of the form $f(t) = 80\ sin(30t)\ N$.

a - Calculate the amplitude of the force transmitted to the ground.

b - To reduce the force transmitted to the ground, do you recommend increasing or decreasing the damping constant of the resilient element ? (Clearly state why)

a-
$m = 200\ kg$
$c = 50\ Ns/m$
$k = 25000\ N/m$
$F_0 = 80\ N$
$\omega = 30\ rad/s$

$$\frac{F_T}{F_0} = \frac{X}{Y} = \sqrt{\frac{k^2 + (c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2}}$$

$$\Rightarrow F_T = 12.92\ N$$

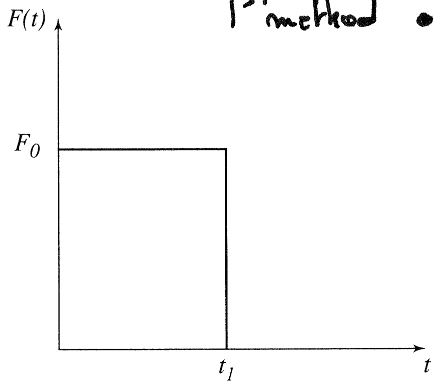
b-

$$r = \frac{\omega}{\omega_n} = \frac{30}{\sqrt{\frac{25000}{m}}} = 2.68 > \sqrt{2}$$

I recommend decreasing the damping because

$$\zeta \searrow \Rightarrow \frac{F_T}{F_0} = \frac{X}{Y} \searrow$$

Problem 3. (30 pts.) Consider an undamped system with mass m and stiffness k subjected to the forcing function in the figure below. If the system starts from rest, determine its response in terms of F_0, t_1, m and k using two different methods. In the first, solve the problem using the traditional convolution integral. In the second method, write the equation of motion of the system subjected to the force in the figure below and solve the differential equation using zero initial conditions.



1st method • $0 \le t \le t_1$ $f(t) = F_0$

$$h(t-\eta) = \frac{1}{m\omega_n}\sin\omega_n(t-\eta)$$

$$x(t) = \int_0^t f(\eta)h(t-\eta)\,d\eta$$

$$x(t) = \frac{F_0}{m\omega_n}\int_0^t \sin\omega_n(t-\eta)\,d\eta$$

$$x(t) = \frac{F_0}{m\omega_n}\left[\frac{1}{\omega_n} - \frac{\cos\omega_n t}{\omega_n}\right] = \frac{F_0}{k}\left[1-\cos\omega_n t\right]$$

• $t > t_1$ $f(t) = 0$

$$x(t) = \int_0^{t_1} f(\eta)h(t-\eta)\,d\eta + \int_{t_1}^{t} \cancel{f(\eta)}^{0}\, h(t-\eta)\,d\eta = \frac{F_0}{k}\left[\cos\omega_n(t-t_1) - \cos\omega_n t\right]$$

2nd method: $m\ddot{x} + kx = F(t)$

• $0 \le t \le t_1$ $m\ddot{x} + kx = F_0 \Rightarrow x(t) = A\cos\omega_n t + B\sin\omega_n t + x_p$

$$m\cancel{\ddot{x}_p}^{0} + kx_p = F_0 \Rightarrow x_p = \frac{F_0}{k} \Rightarrow x(t) = A\cos\omega_n t + B\sin\omega_n t + \frac{F_0}{k}$$

$t = 0$ $\left.\begin{matrix} x_0 = 0 \\ \dot{x}_0 = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} A = -\frac{F_0}{k} \\ B = 0 \end{matrix} \Rightarrow x(t) = \frac{F_0}{k}\left[1 - \cos\omega_n t\right]$

• $t > t_1$ $m\ddot{x} + kx = 0$

$$\Rightarrow x(t) = x_1\cos\omega_n(t-t_1) + \frac{\dot{x}_1}{\omega_n}\sin\omega_n(t-t_1)$$

$$x_1 = \frac{F_0}{k}\left\{1 - \cos\omega_n t_1\right\}$$

$$\dot{x}_1 = \frac{F_0\omega_n}{k}\sin\omega_n t_1$$

Problem 4. (20 pts.) A 200 Kg turbine is mounted on a spring with 8000 N/m stiffness. The turbine has an unbalance of $m \times e = 0.5\ Kg.m$ and runs at a speed of 300 rpm. Determine the total solution describing the turbine vibration assuming zero initial conditions.

$M = 200$ kg

$k = 8000$ N/m

$\omega = \frac{300 \times 2\pi}{60} = 31.416$ rad/s

$\omega_n = \sqrt{\frac{k}{M}} = 6.32$ rad/s

$r = \frac{\omega}{\omega_n} = 4.97 > 1 \Rightarrow \phi = 180^\circ$

$X_{ss} = \frac{me\omega^2}{\sqrt{(k - M\omega^2)^2}} = 0.0026$ m

$x(t) = A\cos\omega_n t + B\sin\omega_n t + X_{ss}\sin(\omega t - \phi)$

$x_0 = 0 = A + X_{ss}\sin(-\phi) \Rightarrow A = 0$

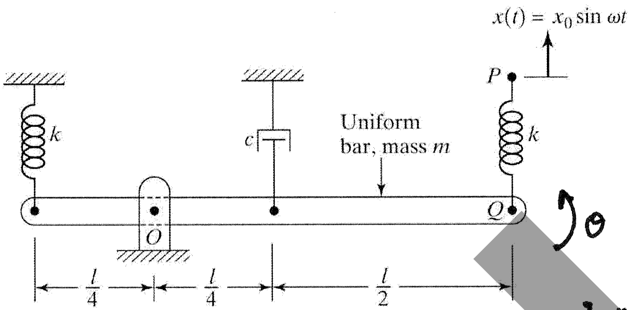
$\dot{x}_0 = 0 = B\omega_n + X_{ss}\omega\cos(-\phi) \Rightarrow B = X_{ss}\frac{\omega}{\omega_n} = 0.0129$ m

$\Rightarrow x(t) = 0.0129\sin(6.32t) + 0.0026\sin(31.416t - \pi)$

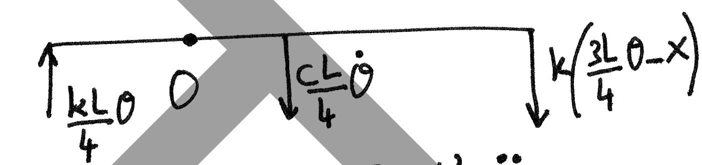
Problem 5. (20 pts.) A uniform bar of mass m is pivoted at point O and supported at the ends by two springs, as shown in the figure below. End P of spring PQ is subjected to a sinusoidal displacement, $x(t) = x_0 \sin(\omega t)$. If the moment of inertia of the bar with respect to O is $J_o = 7mL^2/48$,

a - Derive the equation of motion of the system in terms of θ, the rotation angle about O.

b - Find θ at $t = 20\ s$ in the steady state regime of motion, for $L = 1\ m$, $k = 1000\ N/m$, $c = 500\ Ns/m$, $m = 10\ kg$, $x_0 = 0.01\ m$ and $\omega = 10\ rad/s$.



a-



$$\oplus \circlearrowright \sum M_o = J_o \ddot{\theta} \Rightarrow -\frac{kL^2}{16}\theta - \frac{cL^2}{16}\dot{\theta} - k\left(\frac{3L}{4}\theta - x\right)\frac{3L}{4} = \frac{7mL^2}{48}\ddot{\theta}$$

$$\Rightarrow \frac{7mL^2}{48}\ddot{\theta} + \frac{cL^2}{16}\dot{\theta} + \frac{5L^2}{8}k\theta = \frac{3L}{4}k x_0 \sin \omega t$$

b- $m_{eq} = \frac{7mL^2}{48} = 1.458$ $\quad$ $C_{eq} = \frac{cL^2}{16} = 31.25$

$k_{eq} = \frac{5L^2}{8}k = 625$ $\quad$ $f_{eq} = 7.5$

$$\theta_{ss} = \frac{f_{eq}}{\sqrt{(k_{eq} - m_{eq}\omega^2)^2 + (c_{eq}\omega)^2}} = 0.013 \text{ rad}$$

$$\phi = \tan^{-1}\left[\frac{c_{eq}\omega}{k_{eq} - m_{eq}\omega^2}\right] = 0.578 \text{ rad} \simeq 33.11^\circ$$

$$\theta(t) = \theta_{ss} \sin(\omega t - \phi)$$

$t = 20s$

$\theta = -0.013$ rad